

A CONJECTURE ON PARTITIONS OF GROUPS

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ABSTRACT. We conjecture that every infinite group G can be partitioned into countably many cells $G = \bigcup_{n \in \omega} A_n$ such that $\text{cov}(A_n A_n^{-1}) = |G|$ for each $n \in \omega$. Here $\text{cov}(A) = \min\{|X| : X \subseteq G, G = XA\}$. We confirm this conjecture for each group of regular cardinality and for some groups (in particular, Abelian) of an arbitrary cardinality.

INTRODUCTION

For any finite partition of an infinite group, at last one cell of the partition has a rich combinatorial structure. The Ramsey Theory of groups gives a plenty of concrete examples (see [4], [7]).

On the other hand, the subsets of a group could be classified by their size. For corresponding partition problems see the survey [8].

Given a group G and a subset A of G , we denote

$$\text{cov}(A) = \min\{|X| : X \subseteq G, G = XA\}.$$

The covering number $\text{cov}(A)$ evaluates a size of A inside G and, if A is a subgroup, coincides with the index $|G : A|$.

It is easy to partition each infinite group $G = A_1 \cup A_2$ so that $\text{cov}(A_1)$ and $\text{cov}(A_2)$ are infinite. Moreover, if $|G|$ is regular, there is a partition $G = \bigcup_{\alpha < |G|} H_\alpha$ such that $\text{cov}(G \setminus H_\alpha) = |G|$ for each

$\alpha < |G|$. In particular, there is a partition $G = A_1 \cup A_2$ such that $\text{cov}(A_1) = \text{cov}(A_2) = |G|$. See [5], [9], [10] for these statements, their generalizations and applications.

However, for every $n \in \mathbb{N}$, there is a (minimal) natural number $\Phi(n)$ such that, for every group G and every partition $G = A_1 \cup \dots \cup A_n$, $\text{cov}(A_i A_i^{-1}) \leq \Phi(n)$ for some cell A_i of the partition. It is still an open problem posed in [6, Problem 13.44] whether $\Phi(n) = n$. For the history and results behind this problem see the survey [1].

In [2, Question F], J. Erde asked whether, given a partition \mathcal{P} of an infinite group G such that $|\mathcal{P}| < |G|$, there is $A \in \mathcal{P}$ such that $\text{cov}(AA^{-1})$ is finite. After some simple examples answering this question extremely negatively, we run into the following conjecture.

Conjecture. *Every infinite group G of cardinality κ can be partitioned $G = \bigcup_{n < \omega} A_n$ so that $\text{cov}(A_n A_n^{-1}) = \kappa$ for each $n \in \omega$.*

In this note we confirm Conjecture for every group of regular cardinality and for some groups (in particular, Abelian) of an arbitrary cardinality.

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RESULTS

For a cardinal κ , we denote by $cf(\kappa)$ the cofinality of κ .

Theorem 1. *Let G be an infinite group of cardinality κ . Then there exists a partition $G = \bigcup_{n \in \omega} A_n$ such that $cov(A_n A_n^{-1}) \geq cf(\kappa)$ for each $n \in \omega$.*

Proof. If G is countable, the statement is trivial: take any partition of G into finite subsets. For $\kappa > \aleph_0$, we choose a family $\{G_\alpha : \alpha < \kappa\}$ of subgroups of G such that

- (1) $G_0 = \{e\}$ and $G = \bigcup_{\alpha < \kappa} G_\alpha$, e is the identity of G ;
- (2) $G_\alpha \subset G_\beta$ for all $\alpha < \beta < \kappa$;
- (3) $\bigcup_{\alpha < \beta} G_\alpha = G_\beta$ for each limit ordinal $\beta < \kappa$;
- (4) $|G_\alpha| < \kappa$ for each $\alpha < \kappa$.

Following [11], for each $\alpha < \kappa$, we decompose $G_{\alpha+1} \setminus G_\alpha$ into right cosets by G_α and choose some system X_α of representatives so $G_{\alpha+1} = G_\alpha X_\alpha$. Take an arbitrary element $g \in G \setminus \{e\}$ and choose the smallest subgroup G_α with $g \in G_\alpha$. By (3), $\alpha = \alpha_1 + 1$ for some ordinal $\alpha_1 < \kappa$. Hence $g \in G_{\alpha_1+1} \setminus G_{\alpha_1}$ and there exists $g_1 \in G_{\alpha_1}$, $x_{\alpha_1} \in X_{\alpha_1}$ such that $g = g_1 x_{\alpha_1}$. If $g_1 \neq e$, we choose the ordinal α_2 and elements $g_2 \in G_{\alpha_2+1} \setminus G_{\alpha_2}$ and $x_{\alpha_2} \in X_{\alpha_2}$ such that $g_1 = g_2 x_{\alpha_2}$. Since the set of ordinals $\{\alpha : \alpha < \kappa\}$ is well-ordered, after finite number $s(g)$ of steps, we get the representation

$$g = x_{\alpha_{s(g)}} x_{\alpha_{s(g)-1}} \dots x_{\alpha_2} x_{\alpha_1}, \quad x_{\alpha_i} \in X_{\alpha_i}.$$

We note that this representation is unique and put

$$\gamma_1(g) = \alpha_1, \quad \gamma_2(g) = \alpha_2, \dots, \quad \gamma_{s(g)}(g) = \alpha_{s(g)}, \quad \max(g) = \gamma_1(g).$$

Each ordinal $\alpha < \kappa$ can be written uniquely as $\alpha = \beta + n$ where β is a limit ordinal and $n \in \omega$. We put $f(\alpha) = n$ and denote by $Seq(\omega)$ the set of all finite sequences of elements of ω . Then we define a mapping $\chi : G \setminus \{e\} \rightarrow Seq(\omega)$ by

$$\chi(g) = f(\gamma_{s(g)}(g)) f(\gamma_{s(g)-1}(g)) \dots f(\gamma_2(g)) f(\gamma_1(g)),$$

and, for each $s \in Seq(\omega)$, put $H_s = \chi^{-1}(s)$. Since the set $Seq(\omega)$ is countable, it suffices to prove that $cov(H_s H_s^{-1}) \geq cf \kappa$ for each $s \in Seq(\omega)$.

We take an arbitrary $s \in Seq(\omega)$ and an arbitrary $K \subseteq G$ such that $|K| < cf \kappa$. Then we choose $\gamma < \kappa$ such that $\gamma > \max g$ for each $g \in K$ and $f(\gamma) \notin \{s_1, \dots, s_n\}$. We pick $h \in X_\gamma$ and show that $KH_s \cap hH_s = \emptyset$.

If $g \in KH_s$ and $\gamma_i(g) \geq \gamma$ then $f(\gamma_i(g)) \in \{s_1, \dots, s_n\}$. To prove this we take an arbitrary $g \in KH_s$ and fix $k \in K$, $x \in H_s$ such that $g = kx$. Let $i \in \omega$ be the length of the representation of x after G_γ , which means that there exist $x_1 \in X_{\gamma_1(x)}, \dots, x_i \in X_{\gamma_i(x)}$ such that $x = yx_1 \dots x_i$, for some $y \in G_\gamma$. As $x \in H_s$ it follows that $f(\gamma_j) \in \{s_1, \dots, s_n\}$, for $1 \leq j \leq i$ and since $k \in K \subseteq G_\gamma$ we have that $g = kx = zx_{\alpha_i} \dots x_{\alpha_1}$, where $z \in G_\gamma$. So for any $i \leq s(g)$ for which $\gamma_i(g) \geq \gamma$ we get that $f(\gamma_i(g)) \in \{s_1, \dots, s_n\}$.

Now if $g' \in hH_s$ then the representations of g' and $h^{-1}g'$ after $G_{\gamma+1}$ are equal with the same length i due to the previous argument as $h \in G_{\gamma+1}$. But they are different after G_γ as $f(\gamma_{i+1}(h^{-1}g')) \in \{s_1, \dots, s_n\}$, and then since, $h \notin G_\gamma$, $f(\gamma_{i+1}(g')) = f(\gamma) \notin \{s_1, \dots, s_n\}$.

Hence $KH_s \cap hH_s = \emptyset$, so $h \notin KH_s H_s^{-1}$ and $cov(H_s H_s^{-1})$ can not be less than $cf \kappa$. \square

Theorem 2. *Let λ, κ be infinite cardinals, $\lambda < \kappa$ and let $\{H_\alpha : \alpha < \kappa\}$ be a family of groups such that $|H_\alpha| \leq \lambda$ for each $\alpha < \kappa$. Let G be a subgroup of the direct product $H = \otimes_{\alpha < \kappa} H_\alpha$ such that $|G| = \kappa$. Then there exists a partition $G = \bigcup_{n < \omega} A_n$ such that $\text{cov}(A_n A_n^{-1}) = \kappa$ for each $n \in \omega$.*

Proof. For each $g \in G$, $\text{supt}(g)$ denotes the number of non-identity coordinates of g , $\chi(g) = |\text{supt}(g)|$. For each $h \in \omega$ we put

$$A_n = G \cap \chi^{-1}(n)$$

and show that $\text{cov}(A_n A_n^{-1}) = \kappa$.

We take an arbitrary $K \subset G$ such that $|K| < \kappa$ and denote

$$S = \{\alpha < \kappa : \text{pr}_\alpha g \neq e_\alpha \text{ for some } g \in K\}, \quad T = \kappa \setminus S,$$

$$G_T = G \cap \otimes_{\alpha \in T} H_\alpha.$$

Since $\lambda < \kappa$ and $|G| = \kappa$, we have $|G_T| = \kappa$. If $g \in K A_n A_n^{-1} \cap G_T$ then $|\text{supt}(g)| \leq 2n$. On the other hand, for every $m \in \omega$, there is $h \in G_T$ such that $|\text{supt}(h)| > m$. Hence, $G_T \setminus K A_n A_n^{-1} \neq \emptyset$. \square

Remark 1.

It is well-known [3, Theorems 23.1 and 24.1] that each Abelian group can be embedded into the direct product of countable groups. Applying Theorem 2, we confirm conjecture for Abelian groups. Moreover, if a group G of cardinality κ has an Abelian homomorphic image of cardinality κ (in particular, if G is a free group of rank κ) then Conjecture is valid for G .

Remark 2.

Every infinite group G can be written as a union $G = \bigcup_{\alpha < \text{cf} \kappa} H_\alpha$ of subsets of cardinality $< \kappa$ (and so $\text{cov}(H_\alpha H_\alpha^{-1}) = \kappa$). Hence, Conjecture holds also if $\text{cf} \kappa = \aleph_0$.

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